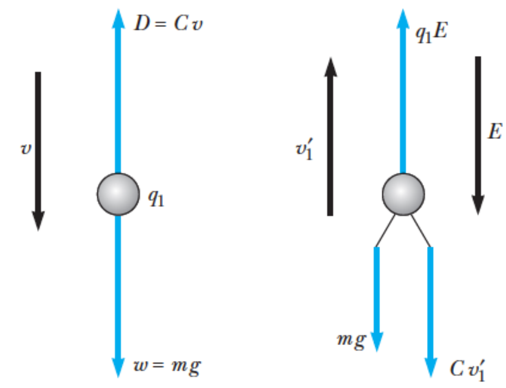
Lecture 1

Question 1. Millikan oil-drop experiment.



(a) Field off (b) Field on

(1) An oil droplet can be viewed as a sphere, and the drag force on a single falling droplet with mass m is given by Stoke's law $D = 6\pi a\eta v$, where a is the droplet radius, η is the viscosity of air, and v is the terminal speed of the droplet. Please show that when the droplet is in equilibrium under the combined action of its weight and the drag force

$$a = \sqrt{\frac{9\eta v}{2\rho g}}$$

where g is the free-fall acceleration (9.81 m s^{-2}) and ρ the mass density of the droplet.

(2) The electric field E is applied to the droplet by charging a parallel-plate capacitor ($E = \Delta U/d$, where ΔU is the potential difference and d the separation between the two plates). Based on the equilibrium condition of the droplet with the electric field E on/off, please show that the charge carried by the droplet q_1 is given by the downward v and upward speeds v_1' as

$$q_1 = \frac{mg}{E}\left(\frac{v + v_1'}{v}\right)$$

(3) Due to the interaction between the charged droplet and air molecules, the charge carried by the droplet undergoes discontinuous change. When the droplet undergoes such a change in its charge, its upward speed becomes v_2'. Please show that the ratio of the charge before and after the change

$$\frac{q_1}{q_2} = \frac{v + v_1'}{v + v_2'}$$

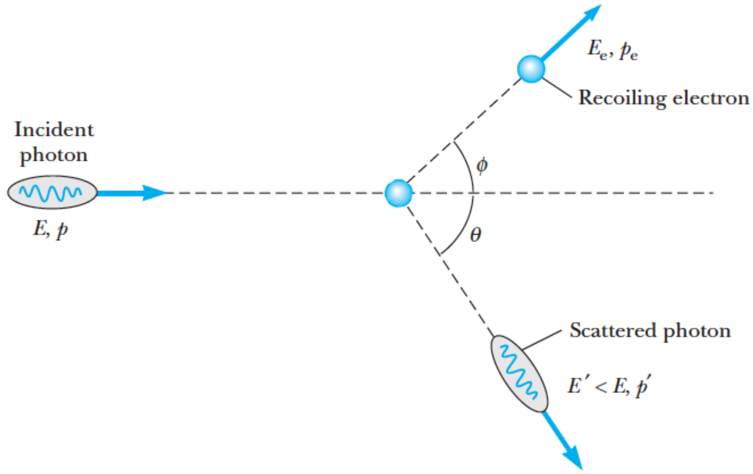
(4) Experimental determination of elementary charge and the proof that charge is quantized: In a Millikan experiment, the distance of rising or falling of a droplet is 0.600 cm, and the average time of fall (field off) is 21.0 s. The observed successive rise times are 46.0, 15.5, 28.1, 12.9, 45.3, and 20.0 s.

(4a) Please prove that charge is quantized by showing that q_1/q_2, q_2/q_3, q_3/q_4, and so on are ratios of small integers, i.e., ratios of whole numbers (e.g., 5/8, 4/3, 2/3, etc.).

(4b) The oil density is 858 kg m^{-3}, and the viscosity of air is 1.83×10^{-5} kg m^{-1} s^{-1}, what is the mass of the oil droplet?

(4c) Calculate the successive charges on the droplet, and from these results determine the value of the elementary charge e. The plate separation is 1.60 cm, and the potential difference is 4550 V for the parallel-plate capacitor.

Question 2. Compton scattering: Hitting an electron at rest by a photon.



From energy conservation $E + m_e c^2 = E' + E_e$ (where c is the speed of light, m_e the electron's mass) and momentum conservations, one could derive the Compton wavelength shift.
(1) Write down the momentum conservation equations – one for the conservation of the horizontal component of the momentum and the other for the vertical component.
(2) Based on momentum conservation, please show that $p_e^2 = (p')^2 + p^2 - 2pp'\cos\theta$
Hint: Put the three momentum vectors $\boldsymbol{p}$, $\boldsymbol{p}'$, and $\boldsymbol{p}_e$ in a triangle. In this way, you will be able to derive the above result easily by using the norm of vector and vector dot product.
(3) Substitute $E = h\nu$ (incident photon) and $E' = h\nu'$ (scattered photon) into the energy conservation relation, and de Broglie wave expression $\lambda = h/p$ into the momentum conservation **(2)**, please derive Compton's result for the increase in a photon's wavelength when it is scatted through an angle θ:

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

Hint: You need to use the expression for the electron's relativistic energy $E_e^2 = p_e^2 c^2 + m_e^2 c^4$
(4) X-rays of wavelength λ = 0.7120 Å are aimed at a block of carbon (graphite). The scattered x-rays are observed at an angle of θ = 45.0° to the incident beam. Calculate the wavelength (in Å) of the scattered x-rays at this angle.
(5) What is the maximum wavelength shift $\Delta\lambda$ possible?
(6) In the actual experiment, the "rest" electrons are the electrons inside graphite. The so-called free electrons in carbon are actually valence electrons with a binding energy of about 4 eV. These free electrons are certainly not at rest, but this binding energy may be ignored for x-rays with λ = 0.712 Å. Why?

Question 3. Blackbody radiation.[1]
(1) Please prove that, at the long-wavelength limit (i.e., $\lambda \to +\infty$), the Planck distribution

$$\varrho^{\mathrm{P}}(\lambda, T) = \frac{8\pi hc}{\lambda^5\left(e^{\frac{hc}{\lambda k_B T}} - 1\right)}$$

goes to the Rayleigh-Jeans distribution $\varrho^{\mathrm{R\text{-}J}}(\lambda, T) = 8\pi k_B T/\lambda^4$, where k_B is the Boltzmann

[1] Some problems are adapted from Atkins, 9th ed, Chapter 7 – Problems.

constant, T the temperature in Kelvin.

(2) Use the Planck distribution $\varrho^{\mathrm{P}}(\lambda, T)$ to deduce the Stefan-Boltzmann law that the total energy density $\mathcal{E}$ (unit: J m^{-3}) of blackbody radiation is proportional to T^4 with the proportionality coefficient $\sigma = 7.56\times10^{-16}$ J m^{-3} K^{-4}. Keep in mind that the total energy density $\mathcal{E}$ sums over the contributions from all wavelengths, i.e., an integral of $\varrho^{\mathrm{P}}(\lambda, T)$ over $\lambda = 0$ to $\lambda = +\infty$.
Useful integral:

$$\int_0^{+\infty} \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$$

(3) An equivalent way to express the Stefan-Boltzmann law is $P = \sigma^{\text{S-B}} T^4$, where P is the total power per unit area emitted at the surface of the blackbody. Power is energy per unit time (unit: Watt = J s^{-1}), and thus P has the unit of Watt m^{-2}. The proportionality coefficient – the Stefan-Boltzmann constant (unit: Watt m^{-2} K^{-4}), $\sigma^{\text{S-B}} = c\sigma/4$, where c is the speed of light and σ the coefficient you deduced in **(2)**.

Estimate the surface temperature of the Sun (approximated as a blackbody): The Sun's radius is given by $R_{\text{sun}} = 7.0\times10^{8}$ m. The average Earth-Sun distance is $R = 1.5\times10^{11}$ m ("average" in the sense that we assume a circular orbit of the Earth). The power per unit area from the Sun is measured at the Earth to be $P_{\text{Earth}} = 1400$ W m^{-2}. What is the surface temperature of the Sun?

Hint: (a) Conservation of radiation energy; (b) Find out the radiation surface areas on the surface of the Sun and on the surface of a large sphere with earth-sun distance as radius.

(4) Please prove that the wavelength $\lambda_{\max}$ corresponding to the maximum in the Planck distribution $\varrho^{\mathrm{P}}(\lambda, T)$ at temperature T obeys Wien's law $\lambda_{\max} T = \text{constant}$, where the value of the constant is 2.898×10^{-3} m K.

Hint: Try to organize the equation about $\lambda_{\max}$ into the form of $xe^x(e^x-1)^{-1}=5$, the numerical solution of which is $x = 4.965$.

(5) In 1792, Thomas Wedgwood, Charles Darwin's relative and a renowned maker of china, observed that all the objects in his ovens, regardless of their chemical nature, size, or shape, became red at the same temperature. Wedgwood's general observation is that objects in his kiln progressed from dull red to orange to colorless as the temperature was raised. Explain his observation qualitatively.

(6) Human eye evolved to be most sensitive at the wavelength of light corresponding to the maximum in the Sun's radiant energy distribution. Our eyes are most sensitive to blue-green light ~ 500 nm. Can you estimate Sun's temperature based on Wien's law? How does this very simple estimation compare to the estimation in **(3)**?

(7) The fraction of the energy density of a star in the visible region of the spectrum determines whether the star will shine. A star that is too small and "cold" is very hard to be found. Suppose a brown dwarf star has a surface temperature of 1000 K. Does this star shine (to our eyes)?

Answer this question by computing the fraction $\mathcal{E}_{\text{visible}}/\mathcal{E}_{\text{total}}$, where $\mathcal{E}_{\text{visible}}$ is the energy density that covers the visible-light region only (integrate from ~ 400 to 720 nm), and $\mathcal{E}_{\text{total}}$ is the total energy density from the star (use the Stefan-Boltzmann law deduced in **(2)**).

Hint: At a high temperature (in the sense that the change of $hc/\lambda k_{\mathrm{B}} T$ is relatively small), the integration range (i.e., from ~ 400 to 720 nm) is small compared to the total range of emitted wavelengths, so we can safely approximate the integrand $\varrho^{\mathrm{P}}(\lambda, T)$ to be a constant equal to the value of the distribution at the mid-point of the visible range (560 nm), i.e., $\varrho^{\mathrm{P}}(\lambda = 560\ \text{nm}, T)$. With this approximation, the value of the integral for computing $\mathcal{E}_{\text{visible}}$ is trivial to evaluate.

Lecture 2

Question 4. Matter waves.
(1) Calculate the wavelength (in Å) of an electron that was accelerated from rest by an electric potential difference of 100 V.
(2) Calculate the speed and the wavelength (in Å) of an electron in a hydrogen atom moving on an orbit with one Bohr radius. Please qualitatively explain why it might be necessary for heavy elements to consider relativistic corrections for the behavior of electrons (and thus their chemistry).
(3) In order to show noticeable wave characters for a tank of Helium (assuming to be ideal gas at all conditions for simplicity), i.e., when the mean free path between two Helium is approximately the same as the wavelength of a single Helium, how low must the temperature be? Let's maintain the gas pressure at 10 atm.
Hint: The average kinetic energy of a monoatomic ideal gas molecule is given by $3k_BT/2$. The mean free path $l = k_B T/\left(4\sqrt{2}\pi r^2 P\right)$, where P is the pressure, $r \sim 0.3$ Å for Helium.

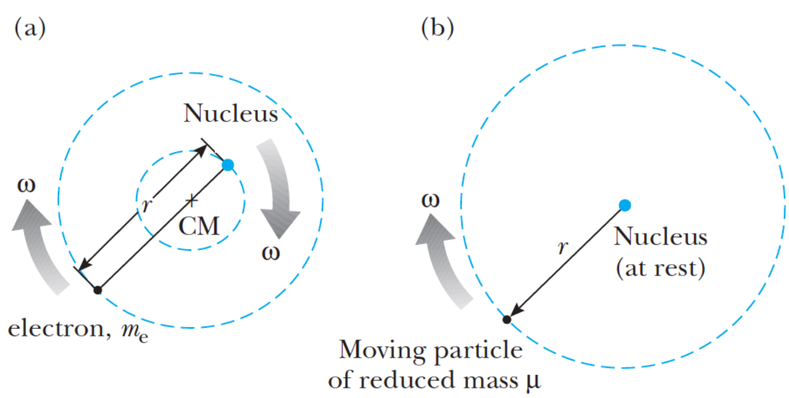
Question 5. Two-body problem and the value of Rydberg constant R_H.
(1) In our derivation of energy levels for Bohr's Hydrogen atom, we set the proton as fixed in the space and the electron is orbiting around it. From the so-obtained energy level expression

$$E_n = -\frac{e^2}{8\pi\epsilon_0 a_0}\left(\frac{1}{n^2}\right)$$

where the Bohr radius

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$$

Compute the value of Rydberg constant R_H for hydrogen to the first decimal ($0.1\ m^{-1}$). What is the percentage error of this value compared to the experimentally measured $R_H = 10967757.6\ m^{-1}$?
Note: The Rydberg constant R_H is one of the most precisely measured physical constants, with a relative standard uncertainty of under 2 parts in 10^{12}.
(2) The above discrepancy, despite tiny, primarily comes from the fact that the hydrogen-atom system is a two-body system, in which the motion of the heaver particle (the proton) shall not be ignored. Figure (a) depicted a two-body system, in which both the electron and the nucleus actually revolve around the center of mass (CM). Figure (b) shows an equivalent single-particle system for considering the effect of nuclear motion, in which the nucleus can be considered to be at rest and m_e is replaced by the reduced mass μ. The single-particle system has an identical angular frequency as the two-body system.



Let us consider a general two-body system, in which the distance between particle A (with mass m_A) and B (with mass m_B) is r.

Please prove that the total rotational kinetic energy of a two-body system is equivalent to the rotational kinetic energy of a single particle with mass $\mu = m_A m_B/(m_A + m_B)$ orbiting around a fixed geometric point O (i.e., the center of mass of the two-body system) with the distance between μ and the point O being r.

Hint: Find out the linear velocities of A and B by considering their rotational radii, respectively. Show that the summation of the kinetic energies ($mv^2/2$) of the two particles can be reduced to a single term, which has the same form as the kinetic energy of a single particle.

(3) The discrepancy between the m_e-based and μ-based computations of R_H would be more severe for, e.g., muonic hydrogen. Muonic hydrogen is a special kind of hydrogen atom in which the electron is replaced by a muon – a particle with the same amount of negative charge but with a mass of 206.77 times heavier than an electron. It has practical applications in muon-catalyzed fusion in nuclear reactors. Please compute the percentage error in R_H for muonic hydrogen.

Question 6. Bohr's historical treatment for Hydrogen-atom energy levels: Correspondence principle.

When Bohr proposed his theory for the Hydrogen atom (in 1913), he did not have the knowledge of de Broglie's matter-wave hypothesis (published in 1923), and thus the derivation we showed in class (which used the concept of standing waves on a ring with de Broglie wavelength) is not how the Hydrogen model was actually developed in the early days of quantum theory (the so-called old quantum theory). The fact that Bohr's original treatment agrees with the standing-wave approach provides good support for de Broglie's hypothesis.

Note: None of the derivations below require the concept of de Broglie wave. They are all classical.

(1) Show that energy E is related to the angular momentum of the electron L by

$$E = -\frac{m_e e^4}{32\pi^2 \varepsilon_0^2 L^2}$$

(2) Next, show that the angular momentum L can be expressed by the electron orbital angular frequency ω and radius r_0

$$L = \frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{\omega r_0}\right)$$

(3) Further, please show that the change in energy dE and the change in angular momentum dL is related by the electron orbital angular frequency ω

$$\frac{dE}{dL} = \omega$$

This is a remarkably simple classical mechanical relation! One step left toward quantization!

(4) Now consider the emission of a photon of energy $dE = h\nu' = \hbar\omega'$ when the electron makes a transition from orbital radius r_1 to r_2. Bohr used the idea of correspondence principle – predictions of quantum theory must correspond to the predictions of classical physics in the region of sizes where classical theory is known to hold. The correct theory must provide a natural transition from the quantum to the classical world. These classical sizes for length, mass, and time are on the order of centimeters, grams, and seconds and typically involve very large quantum numbers. In classical physics, Maxwell's classical electrodynamics theory requires the electron to radiate the light of the same frequency as its orbital motion frequency.

Can you provide a rationalization for angular momentum quantization (which was "derived" based on standing waves and de Broglie wave in the class) based on the above argument?

Lecture 3

Question 7. Complex numbers

(1) Express $z = -1 + i$ in exponential form $z = Ae^{i\theta}$. (Determine A and θ.) Show z on complex plane. Note: There exits an infinite number of θ that gives you the same complex number. One usually chooses the θ that is within the range $-\pi$ to π for such a representation – the principal value of the argument angle.

(2) By using Euler's formula, prove that $\sin\alpha = \frac{(e^{i\alpha} - e^{-i\alpha})}{2i}$ and $\cos\alpha = \frac{(e^{i\alpha} + e^{-i\alpha})}{2}$.

(3) Further, by using the above exponential representations of trigonometry functions, prove that

$$2\sin\alpha\cos\alpha = \sin(2\alpha)$$

and,

$$\frac{d\cos\alpha}{d\alpha} = -\sin\alpha$$

Note: The exponential representation of trigonometry functions is of extremely importance and can be used to simplify many results in math and quantum mechanics significantly.

(4) Prove the following inequalities for complex numbers z_1 and z_2

(a) Triangle inequality: $|z_1| + |z_2| \geq |z_1 + z_2|$

(b) Schwartz inequality (very important for quantum mechanical uncertainty principle):

$$|z_1| \cdot |z_2| \geq |z_1 z_2|$$

Hint: Use the vector representation of a complex number.

Question 8. (Extremely important for the later part of the course!)

The time-independent part of the 1D classical wave equation for free vibration is

$$v^2 \frac{1}{f(x)} \cdot \frac{\partial^2 f(x)}{\partial x^2} = \alpha^2$$

where v is the propagating velocity of the wave, α is a constant. Let us consider the 1D wave with fixed terminal points, i.e., the waves that satisfy the following boundary conditions:

$u(x = 0, t) = f(x = 0)T(t) = 0$ (1st boundary condition at the left terminal point)

and,

$u(x = L, t) = f(x = L)T(t) = 0$ (2nd boundary condition at the right terminal point)

where u is the displacement, L the length of the string.

(1) Show that if $\alpha = 0$, the only solution is $f(x) = 0$, which is not a solution of interest.

(2) Show that if α is a positive real number (i.e., $\alpha > 0$), the solution is not of interest either.

(3) Show that if $\alpha = i\omega$, where ω is a real number, and thus α is a pure imaginary number, the specific solutions are $f(x) = c_1(e^{ikx} - e^{-ikx}) = c\sin(kx)$, in which k is the value of the wave vector, c_1 and c are constants.

(4) Further, by satisfying the 2nd boundary condition, please show that the allowed waves in a 1D box are the ones with the wavelengths $\lambda = 2L/n$, where $n = 1, 2, 3$, etc. These waves are stationary waves, and the wave vector is thus discrete (NOT continuous!). In addition, please explain why we do not include $n = 0, -1, -2, -3$, etc.

(5) Please sketch the first four waves with $n = 1, 2, 3$, and 4, and order them in terms of energy.

Question 9. Using the technique of separation of variables, prove that the wave equation for a 2D freely vibrating membrane can be separated into three independent ODEs, and the time-independent 2D stationary waves can be viewed as a composition of two independent 1D strings.