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[8]In a Book Review in Phys. Today **41**(4), 95 (1988) (of *Einstein's Legacy: The Unity of Space and Time*, Julian Schwinger), C. M. Will writes "...Schwinger's explanation of Einstein's doubling of the deflection of light over the Newtonian value...uses the notion that the equivalence principle alone is sufficient...This idea, first popularized by Leonard Schiff, has since been throroughly discredited. The effect of gravity on rods through space curvature is determined by the field equations of the theory, which are logically independent of the equivalence principle...". In terms of the present treatment, the simple explanation of this alleged Schiff–Schwinger Schwindel is that the equivalence principle correctly generates the radial contraction of space in a perturbation treatment, since it identifies a radial acceleration with a radial force, However, the equivalence principle has nothing to do with what happens to distance intervals at right angles to the force. Since only the radial contraction of space comes into the treatment of the bending of light in lowest order (7), the naive equivalence treatment is justifiably successful in treating the bending of light.
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Elementary examples of adiabatic invariance

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Simple classical one-dimensional systems subject to adiabatic (gradual) perturbations are examined. The first examples are well known: the adiabatic invariance of the product $E\tau$ of energy E and period τ for the simple pendulum and for the simple harmonic oscillator. Next, the adiabatic invariants of the vertical bouncer are found—a ball bouncing elastically from the floor of a rising elevator having slowly varying velocity and acceleration. These examples lead to consideration of adiabatic invariance for one-dimensional systems with potentials of the form $V = ax^n$, with $a = a(t)$ slowly varying in time. Then, the horizontal bouncer is considered—a mass sliding on a smooth floor, bouncing back and forth between two impenetrable walls, one of which is slowly moving. This example is generalized to a particle in a bound state of a general potential with one slowly moving "turning point." Finally, circular motion of a charged particle in a magnetic field slowly varying in time under three different configurations is considered: (a) a free particle in a uniform field; (b) a free particle in a nonuniform "betatron" field; and (c) a particle constrained to a circular orbit in a uniform field.

I. INTRODUCTION

An adiabatic invariant is a physical quantity that remains constant when the parameters of a system are slowly ("adiabatically") varied. For example, a simple pendulum of mass m and string length L may be oscillating with energy E and period τ. If L is gradually decreased, perhaps by pulling the string up through a hole in the ceiling, then E gradually increases and τ decreases, but the product $E\tau$ remains constant. Thus $E\tau$ is said to be an adiabatic invariant for the pendulum.[1] In the "old" quantum theory of Niels Bohr, Max Born, Paul Ehrenfest, Arnold Sommerfeld, and others, before the discovery of the Schrödinger equation, adiabatic invariance played an important role first introduced by Ehrenfest. The classical action S for a periodic system with one degree of freedom x with corresponding canonical momentum p is given by

$$S = \oint p\, dx, \tag{1}$$

where the integral is over one complete cycle of the motion. Ehrenfest, using classical Hamilton–Jacobi theory, proved that S is an adiabatic invariant.[2] He then postulated that S is to be "quantized," i.e., have allowed values given by

$$S = (n + n_0)h, \tag{2}$$

where h is Planck's constant, $n = 0,1,2,3,...$, and n_0 is a constant adjusted to agree with experiment so as to give the correct ground state. The idea was that if a physical quantity is going to make "all or nothing quantum jumps," it should make no jump at all if the system is perturbed gently and adiabatically, and therefore any quantized quantity should be an adiabatic invariant, like S. Equation (2) with $n_0 = \frac{1}{2}$ corresponds to the Bohr–Sommerfeld–Wilson (BSW) quantization rule of the old quantum theory. Nowadays Eqs. (1) and (2) are still useful because they correspond to the (WKB) Wentzel–Kramers–Brillouin approximate solution of the Schrödinger equation.[3] The adiabatic invariance of S is also useful in plasma physics[4] and in accelerator design.[5] Adiabatic invariance is seldom mentioned in undergraduate courses, perhaps because the usual deviation of the adiabatic invariance of S is "advanced," making use of the classical Hamilton–Jacobi theory of mechanics, including canonical transformation to "action and angle" variables.[6]

In this article I consider several simple systems. For each system I find an adiabatic invariant (ad. inv.) from first principles, without invoking the fact that S is an ad. inv. Using the invariant found, I then show that S is indeed an ad. inv. for that system. Then, I generalize these examples to find elementary derivations of the adiabatic invariance of S under two fairly general but rather different kinds of adiabatic perturbation. The last system considered is that of a charged particle moving in a slowly varying magnetic field, where the usual treatment[7] begins with the assumption of the adiabatic invariance of S. Instead, I find the adiabatic invariant directly from Maxwell's equations and, then, as in the other examples, show that S is indeed an adiabatic invariant.

In the end I leave the student no wiser about Hamiltonian–Jacobi mechanics, since I never use it, and give no general proof of the adiabatic invariance of S. Nevertheless, I may convince the student that S is usually an adiabatic invariant, and that adiabatic invariance is interesting, even if not often used in a first course in quantum mechanics.[8]

II. SIMPLE PENDULUM

The pendulum mass m is suspended by a string of length L that passes through a small frictionless hole in the ceiling. One can shorten the string by pulling on the part of the string above the hole. The acceleration component of m along the string (toward the hole) is $L\dot{\theta}^2$, where $\dot{\theta} = d\theta/dt$ and θ is angle of the string with the vertical. The corresponding component of gravitational force is $-mg\cos\theta$. Thus the string tension T is given by $T - mg\cos\theta = mL\dot{\theta}^2$. For small angles, take $\cos\theta = 1 - \theta^2/2$. Then,

$$T = mg - mg\theta^2/2 + mL\dot{\theta}^2. \qquad (3)$$

The kinetic energy K and potential energy V are given by $K = \frac{1}{2}mL^2\dot{\theta}^2$, and $V = -mgL\cos\theta = -mgL + mgL\theta^2/2$. We drop the constant term $-mgL$ from V, giving $V = mgL\theta^2/2$, and write Eq. (3) as

$$T = mg - V/L + 2K/L. \qquad (4)$$

Now, perform a time average of Eq. (4) over one complete cycle. Since L and E are to change only very slowly, we take them as constant over one cycle, and use the well-known results for simple harmonic motion, $V_{av} = K_{av} = E/2$. Then, Eq. (4) becomes

$$T_{av} = mg + E/2L. \qquad (5)$$

Now pull the string from above the hole, gradually shortening the string, width $dL/L \ll 1$ in one cycle. (Note that dL is negative.) The positive work dE done on the pendulum is given by

$$dE = = T_{av}(-dL) = -mg\,dL - E\,dL/2L. \qquad (6)$$

We drop the term $-mg\,dL$, since it just gives the rising of the equilibrium point, corresponding to the term $-mgL$ that we dropped from the potential energy. It has nothing to do with the oscillation of interest. What we call E thus includes only the oscillation energy (kinetic energy and potential energy measured relative to the current equilibrium point.) Then, Eq. (6) becomes $dE/E + dL/2L = 0$, or $d[\ln(EL^{1/2})] = 0$, so that $EL^{1/2}$ is constant, i.e., is an adiabatic invariant:

$$EL^{1/2} = \text{ad. inv.} \qquad (7)$$

Note also that the pendulum period τ is given by $\tau = 2\pi(L/g)^{1/2}$ so that $\tau/L^{1/2}$ is an adiabatic invariant. Multiplying Eq. (7) by the invariant $\tau/L^{1/2}$ gives

$$E\tau = \text{ad. inv.} \qquad (8)$$

for the small-amplitude pendulum. In all of our examples we shall find that $E\tau$ is an adiabatic invariant, and that S is proportional to $E\tau$.

III. ACTION OF THE PENDULUM

The action S is given generally by

$$S = \oint p\,dx = \oint mv\,dx = \oint mvv\,dt$$
$$= \oint 2K\,dt = 2K_{av}\tau, \qquad (9)$$

where K_{av} is the kinetic energy time averaged over one cycle. For the pendulum, or for any simple harmonic oscillator, we have $K_{av} = E/2$. Hence, for an oscillator, Eq. (9) gives

$$S = E\tau. \qquad (10)$$

For the simple pendulum we showed in Eq. (8) that $E\tau$ is an adiabatic invariant; hence, Eq. (10) shows that S is an adiabatic invariant for a simple pendulum undergoing small oscillations.

IV. GENERAL HARMONIC OSCILLATOR

The oscillation energy is

$$E = K + V = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\omega^2x^2. \qquad (11)$$

Let ω vary slowly with time. (No mechanism is specified. It could be that $\omega^2 = g/L$, as in the simple pendulum; but instead of varying L, we could be varying g, as, for example, for a pendulum hung from an elevator ceiling where the elevator acceleration is gradually changing. Or we could have $\omega^2 = k/m$, where k is the spring constant, assumed to be slowly changing.[9]) Over one cycle we can almost neglect the change in ω; hence, E is almost conserved, except for the effect of the nonzero $d\omega/dt$. Thus

$$\frac{dE}{dt} = [m\dot{x}\ddot{x} + m\omega^2x\dot{x}] + m\omega x^2\frac{d\omega}{dt}$$
$$= [\text{zero}] + m\omega x^2\frac{d\omega}{dt}, \qquad (12)$$

where the term [zero] gives the usual energy conservation when ω is constant. Now, time average Eq. (12) over one cycle, using $m\omega x^2_{av} = 2V_{av}/\omega = E/\omega$ so that Eq. (12) becomes $dE/E = d\omega/\omega$, which integrates to give $\ln(E/\omega) = \text{const}$, i.e., $E/\omega = \text{const}$, or since $\tau = 2\pi/\omega$,

$$S = E\tau = \text{ad. inv.} \qquad (13)$$

Quantization of the action for the harmonic oscillation gives, according to Eq. (2), $S = E\tau = (n - \frac{1}{2})h$, $n = 1,2,3,\ldots$. From the perspective of the WKB approximate solution to the Schrödinger equation, we can say that n is the number of de Broglie half-waves contained between the two classical turning points (where $K = 0$), and that $n_0 = -\frac{1}{2} = -\frac{1}{4} - \frac{1}{4}$, where each "$-\frac{1}{4}$" corresponds to penetration of the wavefunction by $\frac{1}{4}$ of a half-wavelength into the two classically forbidden regions.[10] Of course, the exact solution for the oscillator gives the same answer for the quantized energy.

V. THE VERTICAL BOUNCER

A ball bounces elastically from the floor of an elevator, rises, falls, bounces again, etc. First, suppose that the elevator floor is at rest at elevation $x = 0$. The ball bounces up and down with constant energy

$$E = K + V = \tfrac{1}{2}m\dot{x}^2 + mgx = mgh, \tag{14}$$

where $h = gt_1^2/2$ is the maximum height reached by the ball, $t_1 = \tau/2$ is the time to fall from $x = h$ to 0, and τ is the period for one complete round trip. E is constant and equals the kinetic energy at $x = 0$, just after a bounce from the floor.

Now, consider an "adiabatic perturbation" that consists of a slow upward translation of the elevator floor at constant velocity v_0. We are only interested in the energy E that is in the bouncing; i.e., if the ball lies at rest on the floor, its energy increases at the rate of mgv_0, but we set that aside as uninteresting. By transforming to the elevator frame, it is obvious from Galilean invariance that in that frame $E =$ const and $\tau =$ const. They are both adiabatic invariants, under this moving-floor perturbation. (Indeed, they are both exact invariants, even if v_0 is not small.) In the lab frame, what we call the energy E is the lab kinetic energy just after a bounce from the floor. If v is the downward speed in the lab frame just before hitting the floor, then the upward lab speed v' just after the bounce is $v' = v + dv$, where $dv = 2v_0$. (This follows from momentum and energy conservation. The speed of closure just before the collision is $v + v_0$. The speed of separation just after the collision is $v' - v_0$. These must be equal, giving $v' - v = 2v_0$.) Now, the ball rises, falls back, and is about to hit the floor again. But the floor has risen by $dx = v_0\tau$. Thus the ball does not quite recover the complete speed v'. In fact, it falls short of v' by an amount of $dv = 2v_0$, as we let the student show. This $-2v_0$ cancels the $dv = +2v_0$ that the ball gained on the previous bounce, and as a result the ball leaves the floor after this second bounce with the same lab speed v that it left with on the previous bounce. (This result is exact; i.e., it does not depend on v_0 being small compared with v.) Of course, it is leaving from a higher floor, and so it has gained gravitational energy $mgv_0\tau$, but the floor is doing no more time-averaged work on the ball than if the ball were lying at rest on the floor. So, if $E = \tfrac{1}{2}mv^2$ is the lab kinetic energy immediately after any one bounce, we see that $E =$ const. Also ω is constant, as easily seen in the elevator frame. (In the lab frame, we let the student show it.) We see that uniform upward translation of the elevator floor is too trivial a perturbation to be of interest.

Next, let the perturbation consist of a constant upward acceleration of the elevator floor. But in the elevator frame this is equivalent simply to changing the effective gravitational acceleration g, so that we again have the trivial result $E =$ const, $\tau =$ const, and therefore $S =$ const. [We will show that $S = 2E\tau/3$; see Eq. (18).]

To get a "nontrivial" adiabatic perturbation, we must let g vary in time. (Alternatively, the elevator can have nonconstant acceleration, i.e., nonzero "jerk.") We will stay in the elevator frame, since we are only interested in the increase of "bounce energy," not in the increase in energy of a ball lying on the floor. The energy E is given by Eq. (14) with g slowly varying. Then,

$$\frac{dE}{dt} = [m\dot{x}\ddot{x} + mg\dot{x}] + mx\frac{dg}{dt} = [0] + mx\frac{dg}{dt}.$$

Now, time average over one cycle of bouncing from floor, rising, and returning to the floor, giving

$$\frac{dE}{dt} = mgx_{\text{av}}\frac{dg/g}{dt} = V_{\text{av}}\frac{dg/g}{dt}. \tag{15}$$

But, for a bouncing ball, it is very easily shown that $x_{\text{av}} = \tfrac{2}{3}h$, i.e.,

$$V_{\text{av}} = \tfrac{2}{3}E. \tag{16}$$

Then, Eq. (15) becomes $dE/E = \tfrac{2}{3}dg/g$, which integrates to give

$$E/g^{2/3} = \text{ad. inv.} \tag{17}$$

VI. ACTION OF THE VERTICAL BOUNCER

Equation (16) for the bouncer gives $K_{\text{av}} = E - V_{\text{av}} = \tfrac{1}{3}E$. Then, Eq. (9), which is general, gives

$$S = 2K_{\text{av}}\tau = \tfrac{2}{3}E\tau. \tag{18}$$

But $E = mgh = mg[\tfrac{1}{2}g(\tau/2)^2]$, which we solve for τ and insert in Eq. (18) to give

$$S = \tfrac{4}{3}2^{3/2}E^{3/2}/m^{1/2}g. \tag{19}$$

Taking the $\tfrac{2}{3}$ power of Eq. (19) and comparing with Eq. (17), we see that S is indeed an adiabatic invariant.

Carrying out the quantization prescribed by Eq. (2) and using Eq. (19) gives for the bouncer

$$E = (\tfrac{9}{32}mg^2h^2)^{1/3}(n - \tfrac{1}{4})^{2/3}, \quad n = 1,2,3,..., \tag{20}$$

where, from the viewpoint of the WKB approximation, n is the number of half-waves between the two turning points and the constant $n_0 = -\tfrac{1}{4}$ is due to penetration of the wavefunction by $\tfrac{1}{4}$ of a half-wavelength into the upper of the two classically forbidden regions.[10] (At the floor there is no penetration, and thus no contribution to n_0.) Equation (20) agrees quite well, but not exactly, with the exact solution of the Schrödinger equation.[11,12]

VII. ADIABATIC INVARIANT FOR A GENERAL POWER-LAW POTENTIAL[13]

Based on our experience with the harmonic oscillator and the vertical bouncer, we are ready to generalize to all one-dimensional potentials of the form

$$V = ax^n, \tag{21}$$

with slowly varying $a = a(t)$. The energy is

$$E = \tfrac{1}{2}m\dot{x}^2 + ax^n. \tag{22}$$

Under the adiabatic perturbation of $a(t)$, we have $dE/dt = [0] + [(da/a)/dt]V(x)$, i.e.,

$$dE/E = (da/a)V/E. \tag{23}$$

Now, time average over one cycle. Define

$$V_{\text{av}}/E = f. \tag{24}$$

Then, Eq. (23) becomes $d\ln(E/a^f) = 0$, which integrates to give

$$E/a^f = \text{ad. inv.} \tag{25}$$

The virial theorem,[14] or a simple derivation given below, gives

$$2K_{av} = \left(x\frac{dV}{dx}\right)_{av}, \tag{26}$$

and thus for $V = ax^n$ gives $2K_{av} = nV_{av}$, i.e., $2(E - V_{av}) = nV_{av}$, i.e., $V_{av} = 2E/(2+n)$, so that Eq. (24) becomes

$$f = 2/(2+n). \tag{27}$$

(For the harmonic oscillator, $n = 2, f = \frac{1}{2}$, and $a = \frac{1}{2}m\omega^2$. For the vertical bouncer, $n = 1, f = \frac{2}{3}$, and $a = mg$. For the "needle orbit" limit of a classical hydrogen atom with zero angular momentum, i.e., a very eccentric ellipse, $n = 1$, $f = 2$, and $a = -e^2$.)

We now derive Eq. (26). Let the classical turning points be at $x = 0$ and $x = b$. Start by reading Eq. (9) from right to left, then integrate once by parts, noting that the integrated part px is zero at the turning points. Thus

$$2\tau K_{av} = \oint p\,dx = 2\int_0^b p\,dx$$

$$= -2\int_0^b x\,dp = -\oint x\,dp.$$

But $p^2 = 2m(E - V)$, so that $dp = -(m/p)(dV/dx)dx = -(dV/dx)dx/v = -(dV/dx)dt$, and the integral $-\oint x\,dp$ becomes $\oint x(dV/dx)dt$, which equals $\tau(x\,dV/dx)_{av}$, giving Eq. (26).

VIII. PROOF THAT S IS AN ADIABATIC INVARIANT FOR $V=ax^n$

Equation (9) gives $S = 2K_{av}\tau$. But $K_{av} = E - V_{av} = E - fE$, and so

$$S = 2(1-f)E\tau = [2n/(2+n)]E\tau. \tag{28}$$

If we can show that $E\tau$ is an ad. inv., then, since $f = 2/(2+n)$ is a constant, we will have shown that S is an ad. inv. We now calculate τ so as to insert it into Eq. (28). We lump all uninteresting constants, like 2, n, and m, into $\cdots$. Let the two classical turning points be at $x = 0$ and $x = b$. Then,

$$\tau = 2\int_0^b v^{-1}\,dx, \tag{29}$$

or since

$$E = V(b) = ab^n, \tag{30}$$

we have $v = (2/m)^{1/2}(E - V)^{1/2} = \cdots E^{1/2}(1 - V/E)^{1/2} = \cdots E^{1/2}[1 - (x/b)^n]^{1/2}$, so that Eq. (29) becomes, letting $x/b = s$,

$$\tau = \cdots bE^{-1/2}\int_0^b (1 - s^n)^{1/2}\,ds = \cdots bE^{-1/2}, \tag{31}$$

where "$\cdots$" now includes a constant [the integral in Eq. (31)] that depends on n. Multiplying Eq. (31) by E gives

$$E\tau = \cdots bE^{1/2}. \tag{32}$$

But according to Eq. (30), $b = (E/a)^{1/n}$. Thus Eq. (32) becomes

$$E\tau = \cdots E^{(n+2)/2n}/a^{1/n} = \cdots E^{1/nf}/a^{1/n} = \cdots (E/a^f)^{1/fn}, \tag{33}$$

where we used Eq. (27). But, according to Eq. (25), E/a^f is invariant, so that Eq. (33) says that $E\tau$ is invariant. Thus we have shown that S is an adiabatic invariant, when the potential is given by Eq. (21), with $a(t)$ slowly varying.

We have seen that for any power law potential $E\tau$ is an adiabatic invariant, and S is proportional to $E\tau$, as given in Eq. (28).

We now turn to another simple example.

IX. THE HORIZONTAL BOUNCER

A mass m slides on a frictionless horizontal surface, bouncing back and forth between two impenetrable walls. The slowly varying separation between the walls is L. The particle speed is v. The left-hand wall moves very slowly to the right with either constant or slowly varying velocity v_0, with $v_0 \ll v$ so that we can later use calculus. (The right-hand wall does not move.) When mass m bounces elastically from the left-hand wall, its speed v increases by an amount $dv = 2v_0$, as is easily seen by conservation of energy and momentum. Since $E = \frac{1}{2}mv^2$, that gives

$$dE = mv\,dv = 2mvv_0. \tag{34}$$

This increment dE is added once per round trip, i.e., once per time interval $dt = \tau = 2L/v$. Then, $dE/dt = (2mvv_0)(v/2L) = mv^2v_0/L = (2E/L)v_0$. But $v_0 = -dL/dt$. Thus

$$dE/E = -2dL/L. \tag{35}$$

Integration of Eq. (35) gives $\ln(EL^2) =$ const. i.e., for the horizontal bouncer,

$$EL^2 = \text{ad. inv.} \tag{36}$$

The period is $\tau = 2L/v$. Then, $d\tau/\tau = dL/L - dv/v = dL/L - \frac{1}{2}dE/E = 2dL/L$, where we used Eq. (35). This integrates to give

$$\tau/L^2 = \text{ad. inv.} \tag{37}$$

Then, Eqs. (36) and (37) give for the horizontal bouncer

$$E\tau = \text{ad. inv.} \tag{38}$$

X. ACTION OF THE HORIZONTAL BOUNCER

$S = \oint p\,dx = p\oint dx = p2L$. We see that $S =$ ad. inv., since pL is proportional to the square root of the ad. inv. EL^2 given by Eq. (36). Quantization according to Eq. (2) gives $S = nh$, or $p = nh/2L$. Then,

$$E = p^2/2m = n^2h^2/8mL^2, \quad n = 1,2,\ldots. \tag{39}$$

(We also have $S = 2E\tau$, as is easily shown.) The constant n_0 in Eq. (2) is zero according to the WKB approximation since there is no penetration of the wavefunction into either of the two classically forbidden regions. Of course, Eq. (39) also agrees with the exact solution of the Schrödinger equation..

An interesting special case of the horizontal bouncer is found when the wall velocity v_0 is constant. Then, even without the "adiabatic" condition, $v_0 \ll v$, one obtains exact invariance of S, provided one chooses the starting point of the cycle in Eq. (9) to be at the fixed wall.[15]

XI. PROOF THAT S IS AN ADIABATIC INVARIANT UNDER THE "MOVING WALL" PERTURBATION

We now generalize the previous example. We have

$$S = \oint p\,dx = 2\int_a^b p\,dx, \tag{40}$$

where a and b are the left and right turning points. For

$x > a$, let $V(x)$ become a steep, infinite "wall" moving very slowly to the right with constant or slowly varying velocity v_0. That will tend to decrease S by moving the lower integration limit in Eq. (40) to the right. We want $\delta S/dt$. The contribution to $\delta S/dt$ from the moving left turning point (t.p.) in Eq. (40) is

$$(\delta S/dt)_{\text{left t.p.}} = -2p_a\,\delta a/dt = -2p_a v_0. \quad (41)$$

There is another contribution to $\delta S/dt$. Each time the particle hits the left-hand turning point with speed v_a, it gets reflected in an elastic collision, bouncing off with speed $v_a + 2v_0$, with $v_0 \ll v_a$. (We want V to be steep at $x = a$ so that v_a is not zero, and v_0 can then satisfy $v_0 \ll v_a$.) Thus $\delta v_a = 2v_0$. Now, $E = \frac{1}{2}mv^2 + V(x)$. When the particle suffers an impulse from the moving wall at $x = a$, x does not change during the impulse, so that V does not change. Therefore, the increment to the kinetic energy at $x = a$ equals the increment to E. Thus

$$\delta E = \delta K_a = (mv\,\delta v)_a = p_a 2v_0. \quad (42)$$

This δE is conserved after the impulse; i.e., E remains constant during the next round trip, until the next collision. The value of δE tells us how $p(x)$ is changed at each x in the next round trip. We have $p^2 = 2m[E - V(x)]$. Differentiating, holding x constant, gives $2p\,\delta p = 2m\,\delta E$, giving the change in p, at a given x,

$$\delta p = \delta E/v(x).$$

In the action integral this gives an increase in S of

$$\delta S = \oint \delta p\,dx = \delta E \oint dx/v(x)$$

$$= \delta E \oint dt = \partial E\,\tau = 2p_a v_0 \tau, \quad (43)$$

where we used Eq. (42). (We took δE outside the integral, since it is independent of x, and is conserved until the particle returns and hits $x = a$ again.) Since the impulse occurs once per period τ, its contribution to $\delta S/dt$ is $2p_a v_0$. Adding the two contributions, Eqs. (41) and (43), we find

$$\delta S/dt = -2p_a v_0 + 2p_a v_0 = 0. \quad (44)$$

We see that the decrease in S due to the decreased integration path is exactly compensated by the increased momentum! Thus S is an adiabatic invariant under the "slowly moving wall" perturbation.

We note that our earlier example of the vertical bouncer can either be regarded, in the elevator frame, as an arbitrary (but adiabatic) variation of g with time or, in the lab frame, as an arbitrary motion of the elevator floor, which is equivalent to the "slowly moving wall" perturbation.

We now turn to examples involving circular motion.

XII. MASS ON A STRING

Mass m slides on the smooth xy plane and is fastened to a string that passes through a small hole in the plane. We can pull down on the part of the string that is below the plane so as to slowly reduce R, the distance from the hole to the mass. The speed is $v = R\omega$, where $\omega = d\phi/dt$ and ϕ is the azimuthal angle. The energy is $E = \frac{1}{2}mR^2\omega^2$. The angular momentum is $L = mR^2\omega = 2E/\omega$. L is conserved, and thus E/ω is an adiabatic invariant as the string is shortened. The energy $E = L^2/2mR^2$ and angular velocity $\omega = L/mR^2$ both vary like $1/R^2$ as the string is shortened, but their ratio E/ω is conserved. The action is $S = \oint p\,ds = \oint mvr\,d\phi = \oint L\,d\phi = 2\pi L = 2E\tau$. Quantization of the action, $S = nh$, gives $L = nh/2\pi = n\hbar$, and $E = n^2\hbar^2/2mR^2$.

XIII. CHARGED PARTICLE IN A UNIFORM MAGNETIC FIELD THAT CHANGES SLOWLY WITH TIME.[7]

The magnetic field B_z is uniform, and along z. An otherwise free particle of charge q travels in the xy plane in a circular orbit of radius r given by (in mks units)

$$mv = qrB_z, \quad (45)$$

where $m = \gamma m_0$, m_0 is the rest mass, $\gamma = (1-\beta^2)^{-1/2}$, and $\beta = v/c$. The sense of rotation of the particle is such that the magnetic field produced by the "particle–orbit current loop" is along $-B_z$. Now, slowly increase B_z by turning up the magnet current. "Slowly" means $dB_z/B_z \ll 1$ during one period, and thus r is essentially constant during one period. The Faraday induced electric field ϵ is given by

$$\oint \epsilon\,dl = \epsilon(2\pi r) = \pi r^2 \frac{dB_z}{dt}, \quad (46)$$

i.e.,

$$\epsilon = \frac{1}{2} r \frac{dB_z}{dt}. \quad (47)$$

The rate of increase of $mv = qrB_z$ is due to the azimuthal force $q\epsilon$. Thus

$$\frac{d(mv)}{dt} = \frac{d(qrB_z)}{dt} = q\epsilon = \frac{1}{2} qr \frac{dB_z}{dt}, \quad (48)$$

i.e., reading Eq. (48) from right to left,

$$r\,dB_z = 2d(rB_z) = 2r\,dB_z + 2B_z\,dr,$$

i.e.,

$$r\,dB_z + 2B_z\,dr = 0,$$

which integrates to give

$$r^2 B_z = \text{ad. inv.} \quad (49)$$

Thus the flux $\pi r^2 B_z$ enclosed by the particle's orbit is an adiabatic invariant. Alternatively, since

$$(qrB_z)^2 = (mv)^2 = (mv_x)^2 + (mv_y)^2,$$

Eq. (49) is equivalent to

$$[(mv_x)^2 + (mv_y)^2]/B_z = \text{ad. inv.} \quad (50)$$

XIV. GENERALIZATION OF THE INVARIANT FOR A PARTICLE IN A MAGNETIC FIELD

Given a relativistic particle spiraling about the z axis in a *static* magnetic field, produced for example by a resting "bar magnet" aligned along the z axis. The field is cylindrically symmetrical about z, and is mostly along z, near the z axis; but B_z varies slowly with z, so that, since div $\mathbf{B} = 0$, there are small components of B_x and B_y. Since the force $q\mathbf{v}\times\mathbf{B}$ is always perpendicular to $\mathbf{v}$, the particle's kinetic energy $[(cmv)^2 + (m_0c^2)^2]^{1/2} - m_0c^2$ is conserved, so that $(m\mathbf{v})^2$ is conserved, i.e.,

$$(mv_x)^2 + (mv_y)^2 + (mv_z)^2 = \text{const.} \quad (51)$$

Let the particle's v_z be carrying it toward the magnet, so that B_z near the particle is slowly increasing. Now, perform a Lorentz transformation along the z axis to a new inertial frame, where v_z is momentarily zero. Such a transformation leaves mv_x, mv_y, and B_z unchanged, and brings us to the frame where we derived Eq. (50), except that now the reason B_z is increasing is that the magnet is approaching the particle. In that frame, Eq. (50) holds. Now, transform back to the frame where the magnet is at rest. The adiabatic invariant of Eq. (50) is invariant under the Lorentz transformation along z, since mv_x and mv_y are invariant, and so is B_z. Thus, in the magnet rest frame, both expressions (50) and (51) are invariant.

Suppose, for example, the particle starts in a region where (in some units) $B_z = 1$, and where the particle has $(mv_z)^2 = 3$, and $(mv_x)^2 + (mv_y)^2 = 1$, for a total of $(mv)^2 = 4$, which is conserved in what follows. (Let the radius of curvature be $r = 1$.) Suppose the sign of the particle's v_z is taking it into a region with stronger B_z. When B_z has doubled to 2, $(mv_x)^2 + (mv_y)^2$ will also have doubled, to 2, according to Eq. (50), and $(mv_z)^2$ will therefore have decreased to 2, according to Eq. (51). [And r is now 0.707, according to Eq. (49).] When B_z doubles again to 4, $(mv_x)^2 + (mv_y)^2$ will have reached 4, and $(mv_z)^2$ now equals zero. (And $r = \frac{1}{2}$.) This is the turning point, where v_z reverses sign. The spiraling particle reflects there and travels back toward the region of weaker B_z, giving progressively smaller $(mv_x)^2 + (mv_y)^2$, larger $(mv_z)^2$, and larger r, as B_z weakens. Finally, when the particle is at infinity, we have $B_z = 0$, $r = \infty$, $(mv_x)^2 + (mv_y)^2 = 0$, and $(mv_z)^2 = 4$.

Note that we could also get this result by using div $\mathbf{B} = 0$ to give us B_x and B_y and using them to give the force along z. But that is the hard way.

This reflection of a spiraling charged particle away from a region of strong magnetic field is the basis of the "magnetic mirror" approach to confining a hot plasma in a thermonuclear reactor.[7] It is also important in interstellar space. For example, it is the basis of the "Fermi mechanism" by which cosmic ray protons may make elastic "mirror" collisions with interstellar gas clouds carrying trapped magnetic field, giving, after many such random collisions, and according to the theorem of equipartition of energy, huge energies to the cosmic ray protons.[16]

XV. ACTION OF CHARGED PARTICLE IN A MAGNETIC FIELD

We made no use of the action. We consider it now.[7] In a magnetic field

$$S = \oint \mathbf{p}\cdot d\mathbf{l} = p(2\pi r), \tag{52}$$

where p is the magnitude of the canonical momentum $\mathbf{p}$ given by

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}. \tag{53}$$

$\mathbf{A}$ is the vector potential, and $q\mathbf{A}$ is sometimes called the "potential momentum." Without worrying yet about signs, we have

$$\oint m\mathbf{v}\cdot d\mathbf{l} = mv(2\pi r), \tag{54}$$

and for a uniform magnetic field,

$$\oint q\mathbf{A}\cdot d\mathbf{l} = \int q(\mathbf{curl\ A})\cdot d(\mathbf{area}) = q\pi r^2 B_z = mv(\pi r), \tag{55}$$

where in the last step we used Eq. (45). Thus the contribution to S of the potential momentum, given by Eq. (55), has half the magnitude of the contribution of the "mv momentum," given by Eq. (54). Use of the right-hand rule shows that Eq. (55) has the opposite sign to Eq. (54). Thus, in Eq. (53), we need

$$p = qB_z r - \frac{qrB_z}{2} = \frac{qrB_z}{2} = \frac{mv}{2}, \tag{56}$$

so that Eq. (52) gives

$$S = 2\pi rp = \pi rmv = \pi r^2 qB_z. \tag{57}$$

Comparing Eqs. (57) and (49), we see that, indeed, S is an adiabatic invariant.

Since $\tau = 2\pi r/v$, we have $S = \pi rmv = (mv^2/2)\tau$. Quantization of S then gives

$$mv^2/2 = (n + \tfrac{1}{2})h\nu, \quad n = 0,1,2,..., \tag{58}$$

which "looks" nonrelativistic, but is relativistically exact. (Remember that we have $m = \gamma m_0$, and that the cyclotron frequency $\nu = 1/\tau$ is given by $\nu = \nu_0/\gamma$, where $\omega_0 = 2\pi\nu_0 = qB_z/m_0$ is the cyclotron angular frequency at the nonrelativistic limit.) In the nonrelativistic limit, Eq. (58) becomes

$$m_0 v^2/2 = (n + \tfrac{1}{2})h\nu_0, \quad n = 0,1,2,..., \tag{59}$$

which is also the result given by the Schrödinger equation.[17] For a very relativistic particle we can put $v = c$ and $mc^2 = E$, where E is the total energy (including rest energy m_0c^2). Then Eq. (58) gives $E = 2(n + \frac{1}{2})h\nu$. Thus, at the relativistic limit, the quantized energy increases by increments of $2h\nu$, whereas at nonrelativistic energies it increments by $h\nu_0$. Of course, $2h\nu \ll h\nu_0$ for $\gamma \gg 2$.

By comparison of Eqs. (45) and (49), we note that the adiabatic invariance of r^2B_z is equivalent to the invariance of the angular momentum mvr. Why then did we go through the derivation connecting Eqs. (45) and (49)? Why not just say angular momentum mvr is conserved, giving Eq. (49) in one step? (That "derivation" is found in some textbooks.) The answer is that the acceleration given by Faraday's law need not conserve angular momentum mvr, since it is along the particle velocity. Indeed, in the following two examples involving circular motion in a magnetic field, S is an adiabatic invariant, but mvr is not.

XVI. BETATRON ORBIT

In a betatron the magnetic B_{orb} at the particle orbit holds the particle at constant radius $r = R$, while the changing magnetic flux $\pi R^2 B_{\text{av}}$ accelerates the particle (usually an electron) according to Faraday's law, Eq. (48),

$$d(mv) = (qR/2)dB_{\text{av}}. \tag{60}$$

The particle is held at constant radius $r = R$ by B_{orb}, i.e., according to Eq. (45),

$$mv = qRB_{\text{orb}}, \tag{61}$$

i.e., for constant R, but increasing mv,

$$d(mv) = qR\, dB_{\text{orb}}, \tag{62}$$

Comparison of Eqs. (60) and (62) shows that as the particle is accelerated by increasing B_{av} one must increase B_{orb} proportionately, so as to maintain to ratio

$$B_{\text{orb}} = \tfrac{1}{2}B_{\text{av}}. \tag{63}$$

An obvious invariant is obtained from Eq. (61):

$$mv - qRB_{\text{orb}} = 0 = \text{inv}. \quad (64)$$

Since we also have $r = \text{const}$, Eq. (64) is equivalent to

$$mvR - qR^2B_{\text{orb}} = \text{inv}. \quad (65)$$

Of course, the angular momentum mvR is not invariant.

The action is given by

$$S = \oint \mathbf{p}\cdot d\mathbf{l} = p(2\pi R), \quad (66)$$

where p is the magnitude of the canonical momentum $\mathbf{p}$ given by

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}. \quad (67)$$

By comparison of Eqs. (66) and (67) with Eq. (64), we may guess that it turns out that the component A_θ of $\mathbf{A}$ along the particle orbit at the particle orbit is given for the betatron by (without worrying about the sign)

$$A_\theta = RB_{\text{orb}}. \quad (68)$$

We can verify Eq. (68) by considering a simple r dependence for B_z that satisfies Eq. (63). Let

$$B_z = B_{\max}(1 - \tfrac{2}{3}r^2/R^2), \quad (69)$$

where $r = R$ at the orbit. At $r = R$, Eq. (69) gives $B_z = B_{\text{orb}} = B_{\max}/3$. By performing the integral $\int_0^R B_z 2\pi r\,dr = \pi R^2 B_{\text{avg}}$ we verify that $B_{\text{avg}} = 2B_{\text{orb}}$, thus satisfying Eq. (63). The vector potential has only one component:

$$A_\theta = B_{\text{orb}}(\tfrac{3}{2}r - \tfrac{1}{2}r^3/R^2), \quad (70)$$

which satisfies

$$B_z = (\text{curl}\,\mathbf{A})_z = r^{-1}\frac{\partial(rA_\theta)}{\partial r}. \quad (71)$$

We let the reader check that inserting Eq. (70) into (71) gives (69). If we now go to $r = R$ in Eq. (70) we get Eq. (68), as expected. Thus we have, at $r = R$,

$$S = 2\pi(mvR - qR^2B_{\text{orb}}) = 0, \quad (72)$$

which is an invariant according to Eq. (65). Note that if we quantize the action given by Eq. (72) according to Eq. (2), there seems to be only one possible quantum number: $n = n_0 = 0$. I believe that is because the action given by Eq. (72) is not just an adiabatic invariant; it is invariant even if B is changed rapidly. Thus we expect no "quantum jumps" if B is changed rapidly. That is to be contrasted with the free particle in a uniform slowly varying B, where S as given by Eq. (57) is an adiabatic invariant, but would be expected to make quantum jumps if B is varied rapidly enough.

XVII. PARTICLE CONFINED TO A SMOOTH "DOUGHNUT" IN A UNIFORM MAGNETIC FIELD

The charged particle is maintained at constant radius $r = R$ by a constraint force F at the wall of the doughnut, aided by the magnetic field, which is normal to the orbit. Initially, we have constant force F_0, speed v_0, and magnetic field B_0. Then, the field is turned up to B, the velocity increases to v because of Faraday's law, and the constraint force increases to F. Thus, letting positive magnetic field be in the direction such that the constraint force and magnetic force are "helping" one another, we have

$$F_0 + qv_0B_0 = mv_0^2/R, \quad (73)$$

$$F + qvB = mv^2/R, \quad (74)$$

Faraday's law, Eq. (48), gives

$$m(v - v_0) = (qR/2)(B - B_0). \quad (75)$$

By inspection of Eq. (75), we see the obvious invariant

$$mv - (qR/2)B = \text{inv}, \quad (76)$$

or since R is constant,

$$mvR - (qR^2/2)B = \text{inv}. \quad (77)$$

We verify that Eqs. (76) and (77) are equivalent to Eqs. (66) and (67) because, for a uniform field B_z, the vector potential is given by

$$A_\theta = (r/2)B_z, \quad (78)$$

as is verified by inserting Eq. (78) into Eq. (71). Thus S is the expression given by Eq. (77).

There is another invariant in this problem, because of the constraint force. One way to express it is to combine Eqs. (73)–(75) so as to eliminate v and v_0, giving

$$F + \left(\frac{q}{2}\right)^2\left(\frac{R}{m}\right)B^2 = F_0 + \left(\frac{q}{2}\right)^2\left(\frac{R}{m}\right)B_0^2 = \text{inv}. \quad (79)$$

An alternative way to express these results is to introduce the "Larmor precession" angular frequency ω_L given by

$$\omega_L = qB/2m. \quad (80)$$

Let us take $B_0 = 0$, but v_0 not zero, and of course, then, F_0 not zero. Then, the Faraday acceleration of Eq. (75) becomes

$$v - v_0 = R\omega_L, \quad (81)$$

which corresponds to rigid body rotation at the Larmor frequency. That is, if we had a collection of noninteracting particles all with the same q/m (and hence the same ω_L), at various different constrained radii R_i, all orbiting the same central z axis with various velocities v_i of either sign and corresponding orbital angular velocities $\omega_i = v_i R_i$ of various magnitudes and either sign, then after turning on B, each particle would have its orbital velocity incremented according to Eq. (81) by an amount

$$(v - v_0)_i = R_i\omega_L. \quad (82)$$

(This would speed up some of the particles and slow some of them down, depending on their sense of rotation.) The net effect is as if the entire assembly or particles had superposed upon its original motions a rigid body rotation at the Larmor frequency. That is an example of "Larmor's theorem."

The constraint invariant Eq. (79) becomes, for particle i,

$$F_i = F_{0i} - m_i R_i^2 \omega_L^2. \quad (83)$$

We see that the new constraint force F_i needed to hold the orbit at constant R_i equals the old constraint force F_{0i} plus the "centrifugal force" $m_i R_i^2 \omega_L^2$ felt by m_i in a frame rotating at the Larmor frequency. (The magnitude of F_i is either greater than or less than F_{0i}, depending on the sense of rotation of the particle relative to the Larmor rotation.) If the magnetic field is sufficiently weak, we can neglect ω_L^2 (but not ω_L). Then, the centrifugal force term in Eq. (83) is negligible compared with F_{0i}. In that case $F_i = F_{0i}$, and there is then no "desire" of any of the particles to seek new equilibrium radii. That fact is useful in discussing (classically) the diamagnetism of atoms in weak magnetic fields, where the "constraint force" is taken to be the force holding the electrons in their atomic "orbits."[18]

For exquisite modern applications of adiabatic invariance, see Refs. 19 and 20.

[1] Max Born, *Atomic Physics* (Blackie, London, 1946), 4th ed., p. 109, shows simply that $E\tau$ is an adiabatic invariant for the simple pendulum. My derivation is similar to Born's. Lord Rayleigh, Philos. Mag. **3**, 338 (1902) was the first to show that $E\tau$ is an adiabatic invariant for the harmonic oscillator.

[2] Paul Ehrenfest, in *Collected Specific Papers*, edited by M. J. Klein (North-Holland, Amsterdam, 1959).

[3] D. J. Geldart and D. Kiang, Am. J. Phys. **54**, 131–134 (1986) discuss several applications of the Bohr–Sommerfeld–Wilson (BSW) quantization rule and of the Wentzel–Kramers–Brillouin (WKB) approximation based on the Schrödinger equation. This paper also contains numerous references to the BSW and WKB methods.

[4] Subrahmanyan Chandrasekhar, *Plasma Physics* (University of Chicago Press, Chicago, 1960).

[5] Andrei A. Kolomensky and Andrei N. Lebedev, *Theory of Cyclic Accelerators* (North-Holland, Amsterdam; Wiley, New York, 1966), discuss adiabatic invariance in "betatron oscillations" and in "synchrotron oscillations."

[6] Max Born, *The Mechanics of the Atom* (Bell, London, 1927) uses Hamilton–Jacobi formalism to give a general proof that S is an adiabatic invariant. A similar general proof is found in Lev D. Landau and Evgenii M. Lifshitz, *Mechanics* (Pergamon, New York, 1976), 3rd ed., Sec. 41, p. 154.

[7] John David Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., p. 588, assumes, quoting M. Born, Ref. 6, that S is an adiabatic invariant, and thus obtains the results that we discuss.

[8] See, however, Wolfgang Yourgrau and Stanley Mandelstam, *Variational Principles in Dynamics and Quantum Theory* (Pitman, New York, 1960), 2nd ed., p. 109, who discuss Ehrenfest's adiabatic invariance and give a derivation similar to that of Ref. 6.

[9] For a corresponding problem in time-dependent quantum mechanics, see C. F. Lo, "An aging harmonic oscillator," Am. J. Phys. **56**, 827–828 (1988).

[10] See, for example, Frank S. Crawford, "Heuristic derivation of the WKB 'penetration' phase constant for bound states," Am. J. Phys. **56**, 374–375 (1988).

[11] Interesting treatments of the quantum bouncer are given by P. W. Langhoff, Am. J. Phys. **39**, 954–957 (1971); and by R. L. Gibbs, Am. J. Phys. **43**, 25–28 (1975). Further references are given in these two papers.

[12] Frank S. Crawford, "Applications of Bohr's correspondence principle," Am. J. Phys. **57**, 621–628 (1989), discusses the quantum bouncer.

[13] A. M. Cetto and L. de la Pena, "Simple relationship between energy and adiabatic invariants for systems with a power-law potential," Am. J. Phys. **52**, 539–542 (1984), use Hamilton–Jacobi theory to discuss adiabatic invariance for a power-law potential, and give many further references.

[14] Jerry B. Marion, *Classical Dynamics* (Academic, New York, 1970), 2nd ed., p. 233.

[15] Let the particle velocity be v and its distance from the stationary wall be L just after a collision with the moving wall. Just after the next such collision, the velocity is $v' = v + 2v_0$, and the distance is found to be $L' = L(v - v_0)/(v + v_0)$. After the next these become $v'' = v' + 2v_0$, and $L'' = L'(v' - v_0)/(v' + v_0)$, etc. If the cycle in Eq. (9) starts at the stationary wall with velocity v, one finds for the successive values of the action the ratio $S'/S = (v'' + v')L''/(v' + v)L' = 1$. If instead the cycle starts just after leaving the moving wall with velocity v, one finds $S'/S = v(L + L')/v'(L' + L'') = (v + 2v_0)^2(v - v_0)/v^2(v + 3v_0)$, which goes to 1 only in the adiabatic limit $v_0 \ll v$. C. Gignoux and F. Brut, "Adiabatic invariance or scaling?" Am. J. Phys. **57**, 422–428 (1989), use the interesting special case where v_0 is constant and the cycle starts at the stationary wall so that S is exactly conserved to introduce "scaling in space and time" in their study of the motion of a particle confined to a three-dimensional cavity whose dimensions vary linearly with the time. They maintain that a horizontal bouncer with constant v_0 should not be used as a textbook example to demonstrate adiabatic invariance of S, since for v_0 constant one has exact invariance of S without imposing $v_0 \ll v$. Since that is only true if the cycle starts at the fixed wall, their concern seems excessive.

[16] Martin Harwit, *Astrophysical Concepts* (Wiley, New York, 1973), Sec. 6.6.

[17] Lev D. Landau and Evgenii M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1974), p. 151.

[18] Edward M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1985), Berkeley Physics Course, Vol. 2, 2nd ed., p. 417.

[19] Michael Berry, "The geometrical phase," Sci. Am. **259**(6), 46 (1988).

[20] M. Kugler, "Motion in noninertial systems: Theory and demonstrations," Am. J. Phys. **57**, 247–251 (1989), a beautiful discussion of the Foucault pendulum as an example of what everyone except Berry calls the Berry phase (see Ref. 19).

The quantum mechanical Hamiltonian in curvilinear coordinates: A simple derivation

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The quantum mechanical Hamiltonian in curvilinear coordinates is obtained in a straightforward manner without using results from tensor analysis.

I. INTRODUCTION

The transition from the classical to the quantum mechanical Hamiltonian is straightforward when the Hamiltonian is expressed in Cartesian coordinates. Indeed, as everyone knows, it is effected through the substitutions:

$$x \to x, \quad p_x \to -i\hbar(\partial/\partial x).$$

The situation is far more complex when more general,

