Problem Set 2

Note: The problems labeled with a star (★) are somewhat lengthy or tricky in the derivations. You are encouraged but not obligated to solve them for extra credits.

Lecture 4

Question 1. Uncertainty principle.

(1) In the derivation of the uncertainty principle, we have $\langle \Delta F \rangle^2 \langle \Delta G \rangle^2 \geq \left(\frac{i}{2} \langle [\hat{F}, \hat{G}] \rangle \right)^2$.

Please prove that the term $\frac{i}{2} \langle [\hat{F}, \hat{G}] \rangle$ is always a real number. (Hint: Show that the commutator itself is an anti-Hermitian operator, and its expectation value is purely imaginary.)

(2) Define $\Delta E = \Delta H$ and $\Delta t = \frac{\Delta Q}{|d \langle Q \rangle / dt|}$ (in which Q is any physical observable, and its corresponding operator does not explicitly depend on time), please prove the energy-time uncertainty principle: $\Delta E \Delta t \geq \hbar/2$. Δt characterizes the amount of time it takes that expectation value of Q to change by one standard deviation.

Hint: Try to evaluate $\Delta\hat{H}\Delta\hat{Q}$ first, which requires $[\hat{H}, \hat{Q}]$. The Ehrenfest equation of motion links $[\hat{H}, \hat{Q}]$ with $d\langle\hat{Q}\rangle/dt$.

(3) Compute $\Delta x \Delta p_x$ for the wave function $\psi(x) = \frac{1}{x^2 + a^2}$ ($-\infty < x < +\infty$), and validate the uncertainty principle in this case. Remember to normalize the wave function first.

Useful integrals: $\int_0^{+\infty} \frac{1}{(x^2+a^2)^3} dx = \frac{3\pi}{16a^5}$ $\quad \int_0^{+\infty} \frac{1}{(x^2+a^2)^2} dx = \frac{\pi}{4a^3}$

$$\int_0^{+\infty} \frac{x^2}{(x^2+a^2)^4} dx = \frac{\pi}{32a^5} \qquad \int_0^{+\infty} \frac{x^2}{(x^2+a^2)^2} dx = \frac{\pi}{4a}$$

Question 2. Consider a particle with mass m moving in one-dimensional space, with the Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$. The associated energy eigenvalues are known to be E_n.

(1) If $V(x) = C \ln(\frac{x}{x_0})$, where C and x_0 are two constants, please compute the expectation value

of kinetic energy for the stationary states by using the virial theorem.

(2) Now let us consider another Hamiltonian $\hat{H}_\lambda$, which is a modification of $\hat{H}$: $\hat{H}_\lambda = \hat{H} + \frac{\lambda \hat{p}}{m}$

(2a) Please use the Ehrenfest equation of motion to prove that for the stationary states governed by $\hat{H}_\lambda$, the expectation value of the momentum is equal to $-\lambda$.

(2b) Please use the Hellmann-Feynman theorem to solve for the energy eigenvalues for the stationary states governed by $\hat{H}_\lambda$.

(3) For the stationary states governed by the general Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$, let us define x_0 as the expectation value of the position, and p_0 as the expectation value of the momentum. Please use the Ehrenfest theorem to prove:

(3a) If the potential $V(x)$ is quadratic around x_0, that is, if we can truncate the Taylor expansion of $V(x)$ to the second order at $x = x_0$, then $\frac{d}{dt} p_0 = -\frac{dV(x_0)}{dx_0}$, which has exactly the same form as in classical mechanics.

(3b) In general, the potential $V(x)$ is not quadratic. If we are able to partition the potential into two parts, i.e., $V(x) = V_{\text{I}}(x) + V_{\text{II}}(x)$, then we have $\frac{d}{dt} p_0 = -\frac{dV_{\text{I}}(x_0)}{dx_0} + \left\langle -\frac{dV_{\text{II}}(x)}{dx} \right\rangle$. The physical meaning of this result is that, unlike a classical particle, which responds only to the force at x_0, the quantum particle responds to the force (generating by V_{II}) at neighboring points as well.

Lecture 5

Question 3. Consider the following operators:

$$\boldsymbol{L}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{L}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(1) What are the possible values one can obtain if L_z is measured?

(2) Take the state in which $L_z = 1$. In this state, what are $\langle L_x \rangle$, $\langle L_x^2 \rangle$, and ΔL_x?

(3) Find the normalized eigenstates and the eigenvalues of L_x in L_z basis. (Express each L_x eigenstate in terms of the *normalized* L_z eigenstates.)

(4) If the particle is in the state with $L_z = -1$, and L_x is measured, what are the possible outcomes and their probabilities?

(5) Consider the state $|n\rangle = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}$ in the L_z basis. If L_z^2 is measured in this state and a result +1 is obtained, what is the state after the measurement? How probable was this result?

Question 4. Given the following two-level system:

$$\hat{H} = \varepsilon\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right)$$

where $|1\rangle$ and $|2\rangle$ are orthonormal basis and ε is a number with the dimensions of energy. Find the energy levels and eigenstates of this system.

Question 5. Consider a particle (ignore its orbital motion) with the following spin matrices:

$$\vec{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\vec{e}_x \qquad \vec{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\vec{e}_y \qquad \vec{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\vec{e}_z$$

where $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are the three unit vectors along the x, y, and z axis. We apply an external static magnetic field $\vec{B} = B(\vec{e}_x + \vec{e}_y + \vec{e}_z)$ on this particle.

The Hamiltonian for the system is $\hat{H} = -\mu\vec{B}\cdot\vec{S}$, where μ is a positive physical constant. At $t = 0$, this particle is in the eigenstate of S_z, i.e., $\vec{S}(t=0) = \frac{\hbar}{2}\vec{e}_z$. What is the periodicity T for the total spin $\vec{S}$ to flip back to $\vec{S}(t=T) = \frac{\hbar}{2}\vec{e}_z$?

Question 6. 1D Harmonic oscillator:

(1) Prove that the rising operator $\left(\hat{a}^+\right)^2$ increases the energy of the eigenstate $|n\rangle$ by $2\hbar\omega$.

(2) Evaluate $\langle n|x|n\rangle$, $\langle n|x^2|n\rangle$, $\langle n|\hat{p}_x|n\rangle$, $\langle n|\hat{p}_x^2|n\rangle$ and $\Delta x\Delta p_x$.

Lecture 6

Question 7. Consider a 1D free particle described by the following Gaussian wave packet:

$\Psi(x,t=0)=A\exp\left[ik_0(x-x_0)-\left(\dfrac{x-x_0}{2a}\right)^2\right]$, in which $a>0$ and A is the normalization constant.

(1) Compute the normalization constant A.

(2) Compute the expectation value of x and x^2 at $t = 0$.

(3) Write down the corresponding momentum (k) representation $\varphi(k)$, and compute the expectation value $\langle\varphi(k)|k|\varphi(k)\rangle$ and $\langle\varphi(k)|k^2|\varphi(k)\rangle$. Do they change with time?

(4) What is the value of $\Delta x\Delta p$ at $t = 0$?

(5) Compare the form of $|\Psi(x,t=0)|^2$ and $|\Psi(x,t)|^2$, what is $\Delta x(t)$?

Useful integral: $\displaystyle\int_{-\infty}^{+\infty}\exp\left[-\left(\alpha^2x^2+i\beta x+i\gamma x^2\right)\right]dx=\left(\frac{\pi}{\alpha^2+i\gamma}\right)^{1/2}\exp\left[-\frac{\beta^2(\alpha^2-i\gamma)}{4(\alpha^4+\gamma^2)}\right]$

Question 8. Let us consider 1D systems. In the momentum representation, we have the following rules: (a) $\hat{p}|p\rangle=p|p\rangle$, where $|p\rangle$ is the momentum basis. Note that in terms of p, the normalized planewave is $|p\rangle=\dfrac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$; (b) Orthonormalization: $\langle p'|p\rangle=\delta(p-p')$, in which $\delta(p-p')$ is the Dirac function.

(1) Write down the expression for the coordinate operator $\hat{x}$ in the momentum space, based on the canonical quantization condition $[\hat{x},\hat{p}]=i\hbar$.

(2) Let us denote the system quantum state as $|\psi\rangle$, and the energy eigenequation is $\hat{H}|\psi\rangle=E|\psi\rangle$.

The eigenfunctions in momentum space $\varphi(p)$ are the projection of $|\psi\rangle$ in $|p\rangle$: $\varphi(p) \equiv \langle p|\psi\rangle$.

Write down the energy eigenequation in the momentum representation and the expression for the matrix element $\langle p|V|p'\rangle$.

★(3) A deep attractive potential generated by a point at $x = 0$ can be modeled by the Dirac delta function potential $V(x) = -V_0\delta(x)$, where V_0 is a constant (with the unit of energy).

Using the energy eigenequation in the momentum representation, solve for the energy eigenvalues for the bound states (*i.e.*, $E < 0$). Useful integral: $\int_{-\infty}^{+\infty} \frac{1}{x^2+\alpha^2}dx = \frac{\pi}{\alpha}$

<u>Lecture 7</u>

Question 9. A particle with mass m in 3D is governed by: $\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} + \lambda r^3$, in which the constant $\lambda > 0$, $\boldsymbol{p}$ is the total linear momentum, and r is the radial distance from the origin. Please separate the variables in the energy eigenequation using the spherical coordinates (r,θ,φ), i.e., write down three eigenequations, each of which only depends on one of the spherical coordinates.

Question 10. Prove that in 1D, if the potential function is an even function, then the wavefunctions of the eigenstates must be either odd or even.

<u>Lecture 8</u>

Question 11. Consider a particle with mass m in a 1D finite well, in which the potential $V(x)$:

$$V(x) = \begin{cases} 0 & (|x| > \frac{L}{2}) \\ -V_0 & (-\frac{L}{2} \le x \le \frac{L}{2}) \end{cases}$$

where V_0 is a positive constant, and L is the length of the box.

(1) For the bound states ($E < 0$), show that the energy levels satisfy $\alpha = \beta\tan(\beta L/2)$ (for the *even* $\psi(x)$ in $-\frac{L}{2} \le x \le \frac{L}{2}$) or $\alpha = -\beta\cot(\beta L/2)$ (for the *odd* $\psi(x)$ in $-\frac{L}{2} \le x \le \frac{L}{2}$), where

$$\alpha = \frac{\sqrt{-2mE}}{\hbar} \quad \text{and} \quad \beta = \frac{\sqrt{2m(E+V_0)}}{\hbar}.$$

Hint: Use the statement in Question 10 to construct your wave functions with proper symmetry.

★(2) Are you guaranteed to have bound states in the well for any positive value of V_0 ($V_0 > 0$)?

(3) For the scattering states (the particle with the energy $E > 0$ is shooting from the left side of the potential), show that in general the transmission probability $T \leq 1$.

$T < 1$ is called the *non-classical reflection*. (Imagine that you stand at the cliff and throw a ball, the ball will partially be bounced back without hitting anything.) What does the energy E need to be (the *resonance* energy) in order to let the well become perfectly transparent, i.e., $T = 1$?